Constructing morphisms by diagram chases

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University of Siegen

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Outline

Outline

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Section 1

Classical diagram chases

What are diagram chases?

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of morphisms

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Diagram chases are a tool in homological algebra used for proving

1. properties
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of morphisms situated in (commutative) diagrams of prescribed shape.

Example: deducing a property

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Consider the following commutative diagram of abelian groups:

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$$\begin{array}{ccccccc}
A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
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\end{array}$$

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Consider the following commutative diagram of abelian groups:

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If

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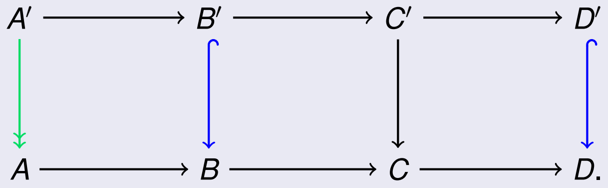

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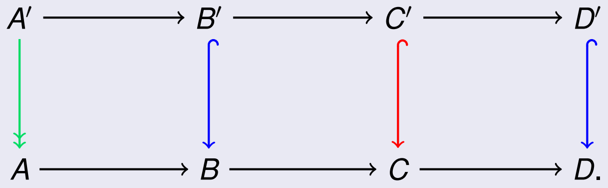

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then the red map is injective.

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Claim: There exists a morphism making the left square commutative.

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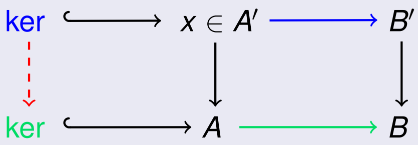

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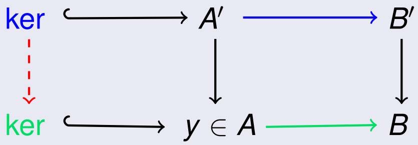

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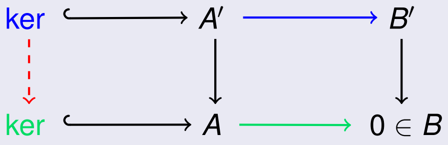

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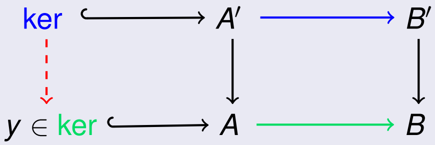

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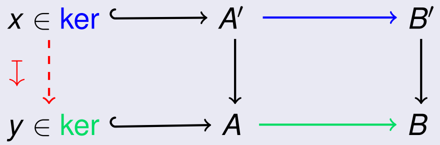

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Because we were working with abelian groups, we were able to use **elements** of their underlying sets. What if we don't have elements?

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$$\text{finite dimensional vector spaces} \longleftrightarrow \mathbb{N}_0$$

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$$\ker\varphi \xhookrightarrow{\kappa} A \xrightarrow{\varphi} B$$

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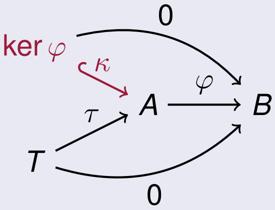

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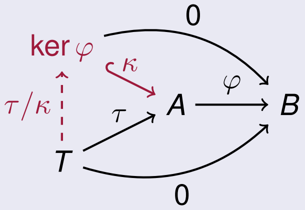

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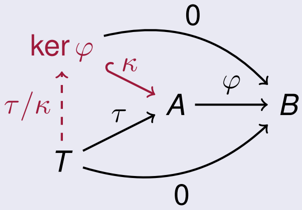

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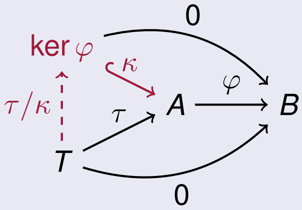

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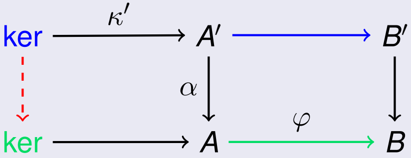
- KernelLift$(\varphi, \tau) = \tau/\kappa$

Example: existence of a morphism



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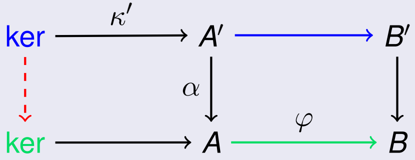

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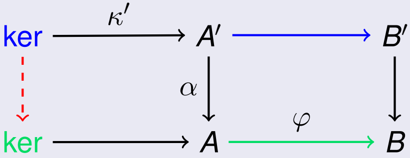

$\downarrow =$

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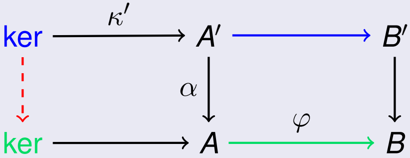

⇣ $= \qquad \alpha \circ \kappa'$

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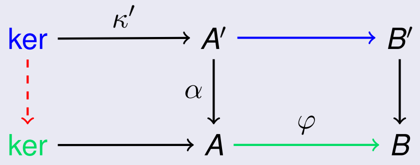

$\dashrightarrow = \text{KernelLift}(\varphi, \alpha \circ \kappa')$

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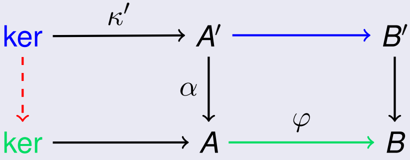

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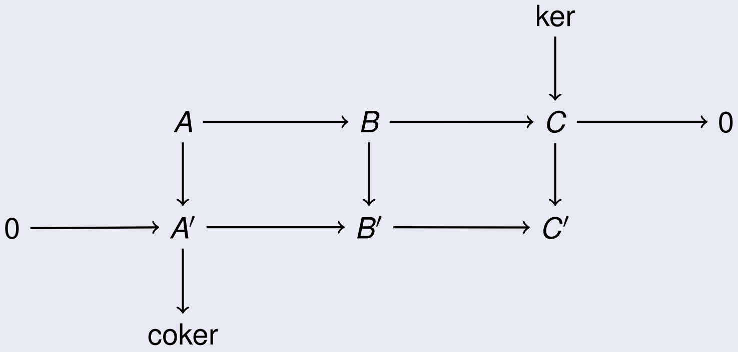
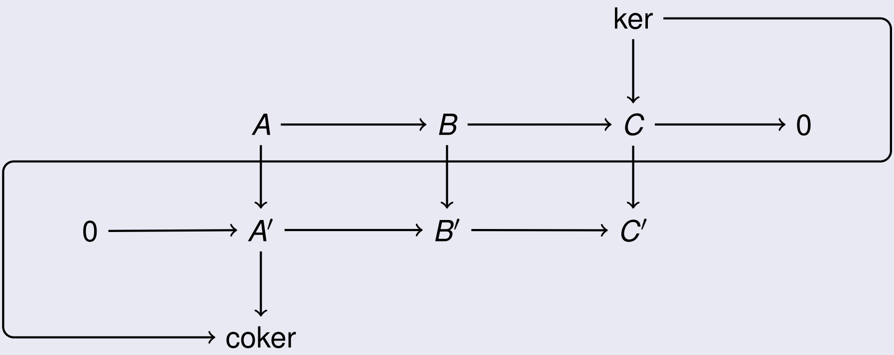
What do we do when the diagrams become larger?

A larger diagram



A larger diagram



An arbitrarily large diagram

$$
\begin{array}{ccccccccccc}
\cdots & & \cdots & & \cdots & & \cdots & & \cdots & & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \to & C^{p,q+1} & \longrightarrow & C^{p+1,q+1} & \longrightarrow & C^{p+2,q+1} & \longrightarrow & C^{p+3,q+1} & \to & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
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The group valued embedding theorem (Mitchell)

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Any small abelian category $\mathbf{A}$ admits an exact covariant embedding

$$F : \mathbf{A} \hookrightarrow \mathbf{Ab}$$

into the category of abelian groups.

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$$\mathrm{Hom}_{\mathbf{A}}(A, B) \qquad \subseteq \qquad \mathrm{Hom}_{\mathbf{Ab}}(FA, FB)$$

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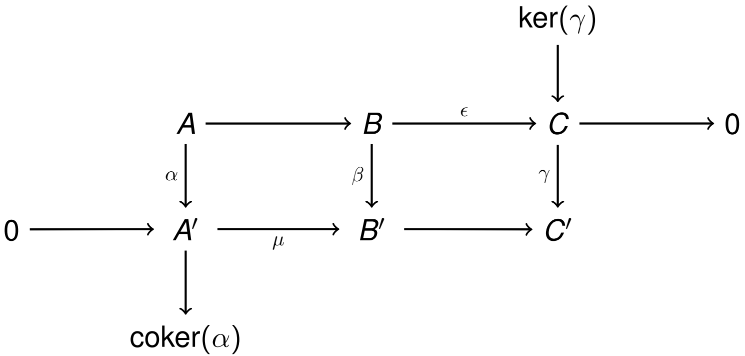
Problem: This isomorphism between Hom-sets is **not constructive.**

Section 2

Constructive diagram chases

Connecting homomorphism in the snake lemma



Wanted: $\ker(\gamma) \xrightarrow{\partial} \operatorname{coker}(\alpha)$.

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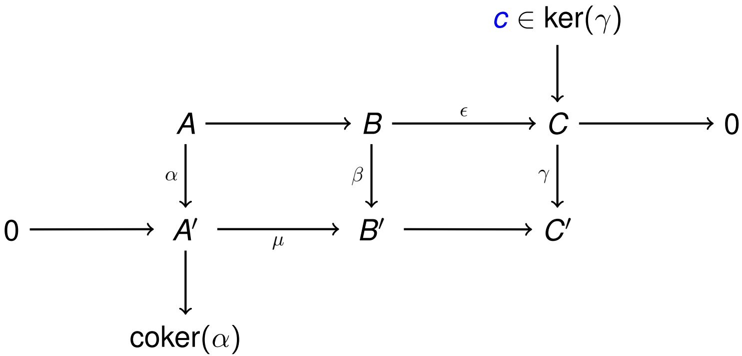

Start: $c \in \ker(\gamma)$.

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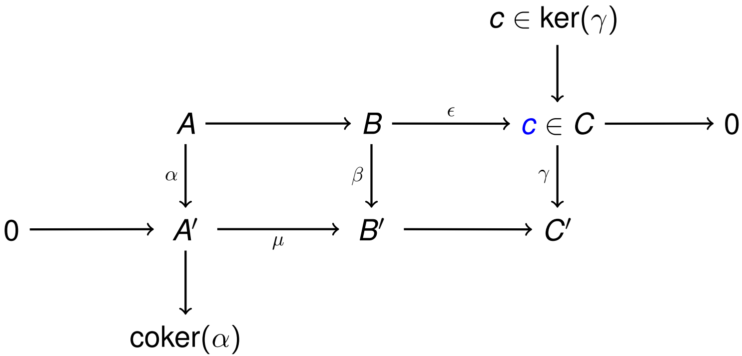

This lies in C.

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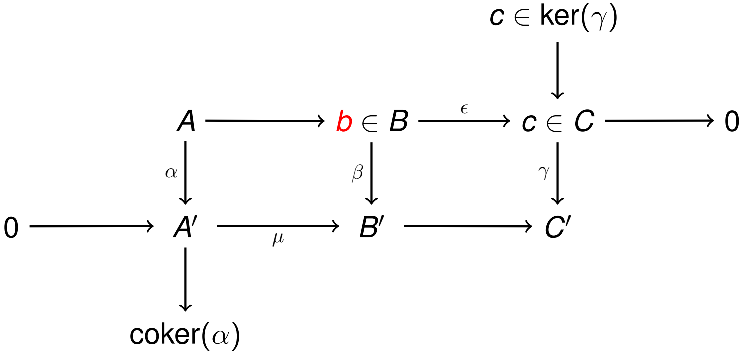

Choose: $b \in \epsilon^{-1}(\{c\})$.

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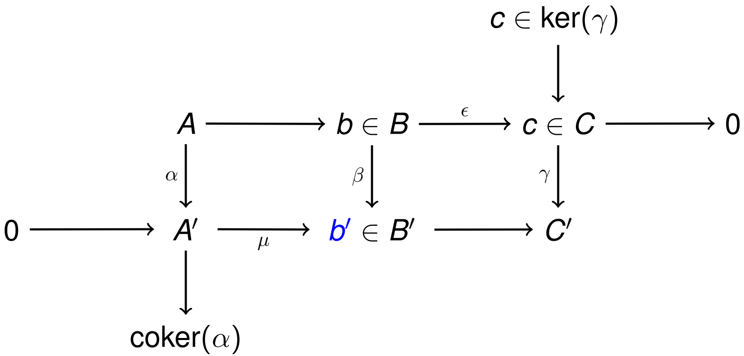

Map: $b \overset{\beta}{\mapsto} b'$.

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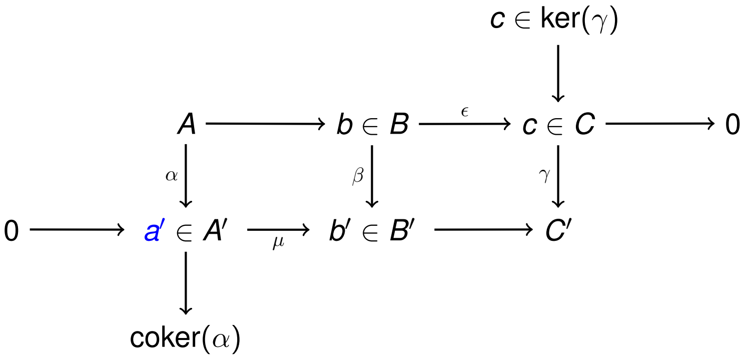
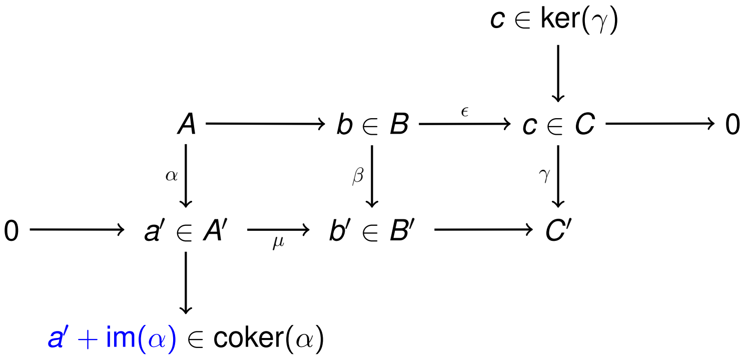


Compute: $a' \in \mu^{-1}(b')$.

Connecting homomorphism in the snake lemma



Map: $a' \mapsto a' + \mathrm{im}(\alpha)$.

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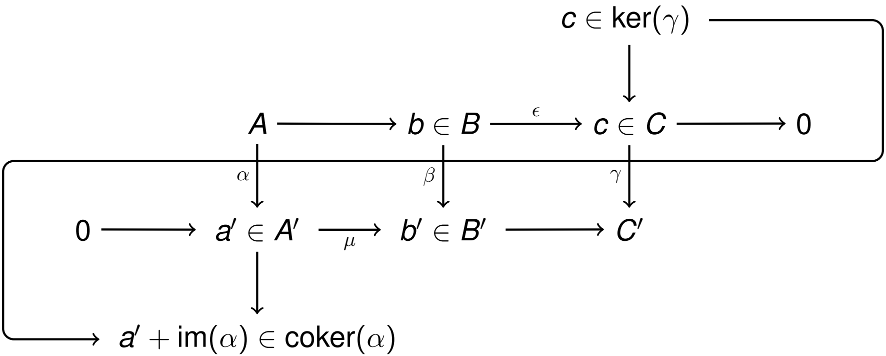

Result: $c \overset{\partial}{\mapsto} a' + \mathrm{im}(\alpha)$.

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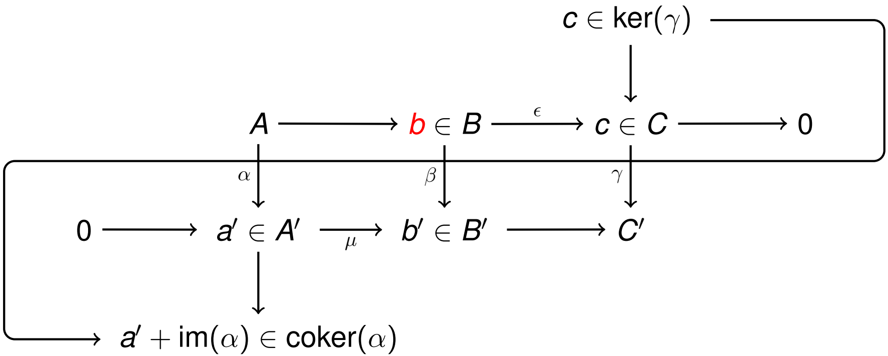
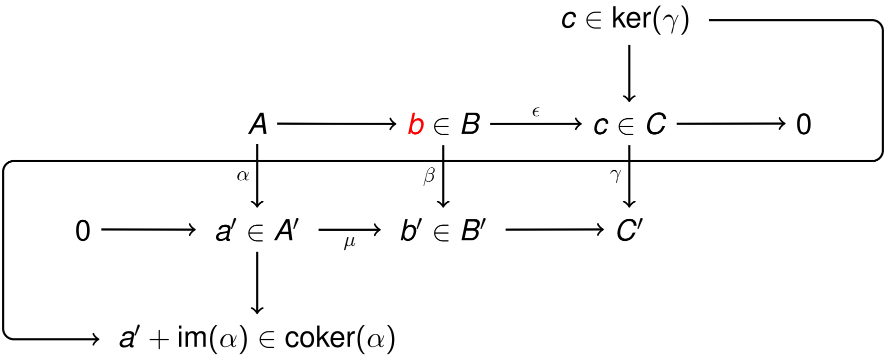


Result: $c \overset{\partial}{\mapsto} a' + \mathrm{im}(\alpha)$. Independent of the **choice**.

Connecting homomorphism in the snake lemma



Idea: use **relations** instead of maps. $c \mapsto \epsilon^{-1}(\{c\})$

Relations

Let A, B be abelian groups.

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Definition

A subgroup $f \subseteq A \oplus B$ is called a **relation from A to B.**

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is a relation from C to B, called **pseudo-inverse of** ϵ.

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Composition of relations

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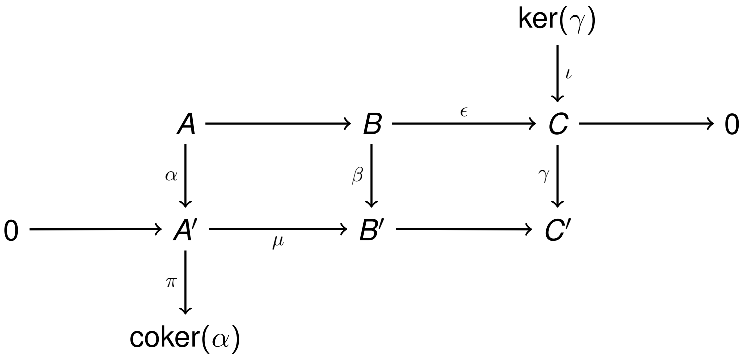
$$g \circ f := \{(a, c) \in A \oplus C \mid \exists b \in B : (a, b) \in f, (b, c) \in g\}$$

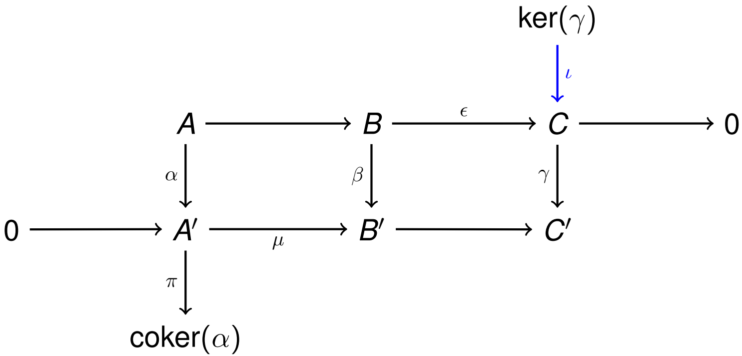
If f and g correspond to maps, this describes their usual composition.

Snake lemma revisited



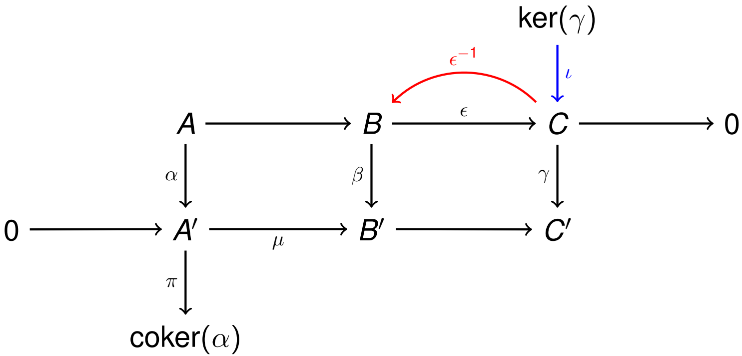
Wanted: $\ker(\gamma) \xrightarrow{\partial} \operatorname{coker}(\alpha)$.

Snake lemma revisited



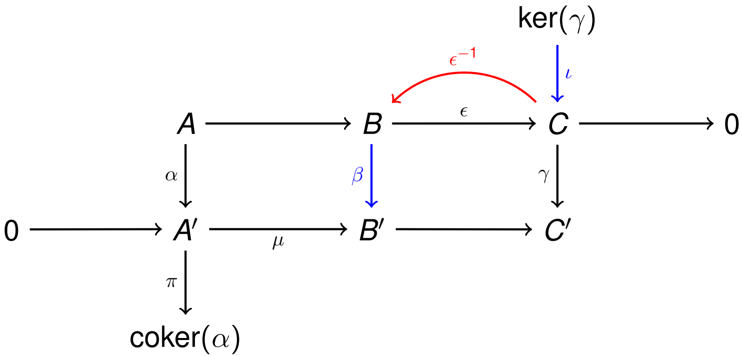
ι

Snake lemma revisited



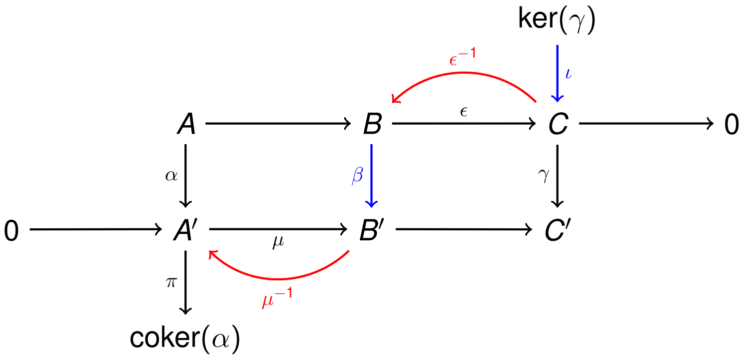
$\epsilon^{-1} \circ \iota$

Snake lemma revisited



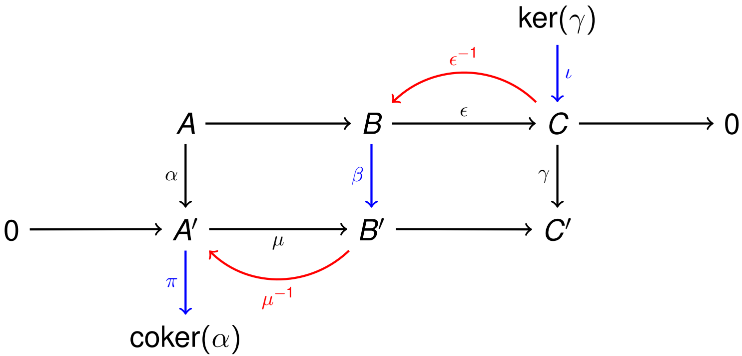
$$\beta \circ \epsilon^{-1} \circ \iota$$

Snake lemma revisited



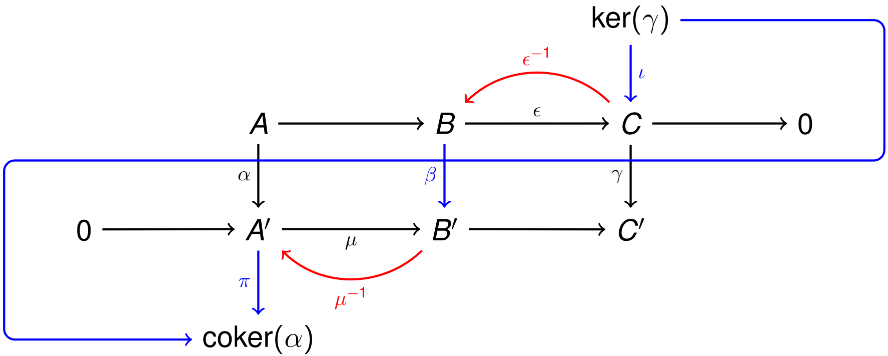
$$\mu^{-1} \circ \beta \circ \epsilon^{-1} \circ \iota$$

Snake lemma revisited



$$\pi \circ \mu^{-1} \circ \beta \circ \epsilon^{-1} \circ \iota$$

Snake lemma revisited



∂ is a composition of relations!

From relations to generalized morphisms

- **Wanted**: a categorical framework for relations.

From relations to generalized morphisms

- **Wanted**: a categorical framework for relations.
- **Solution**: generalized morphisms.

From relations to generalized morphisms

Let A, B be objects in an abelian category **A**.

From relations to generalized morphisms

Let A, B be objects in an abelian category $\mathbf{A}$.

Relation

$$\begin{array}{c} A \oplus B \\ \uparrow \\ D \end{array}$$

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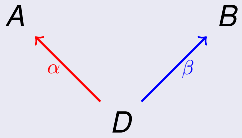

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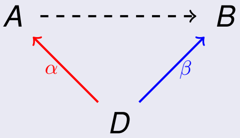

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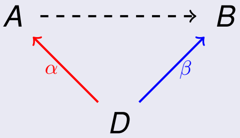

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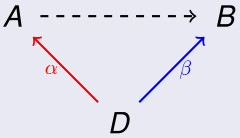

Equality

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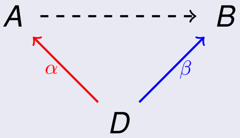

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Two spans (α, β) and (α', β') are **equal as generalized morphisms** if

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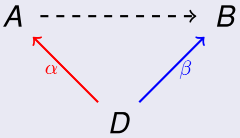


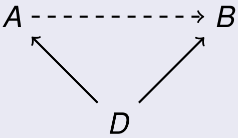
Equality

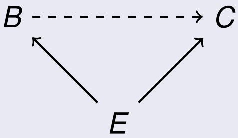
Two spans (α, β) and (α', β') are **equal as generalized morphisms** if

$$\operatorname{im}\left((\alpha, \beta) : D \to A \oplus B\right) = \operatorname{im}\left((\alpha', \beta') : D' \to A \oplus B\right).$$

Composition of generalized morphisms

Composition





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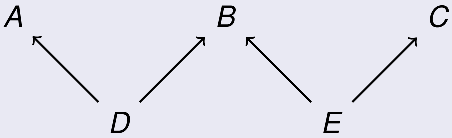

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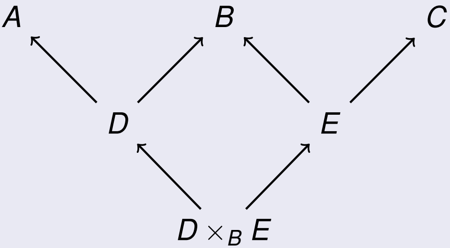

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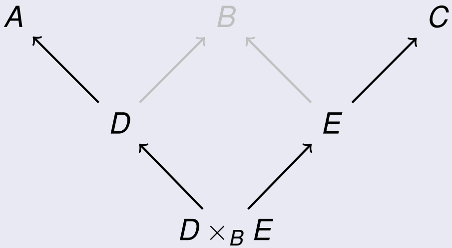

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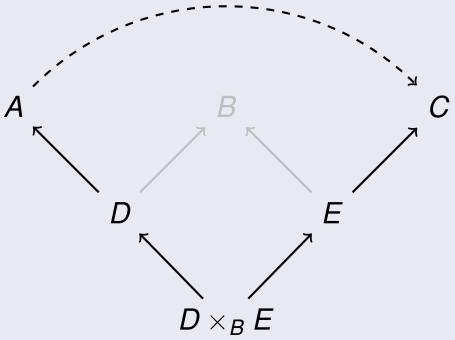

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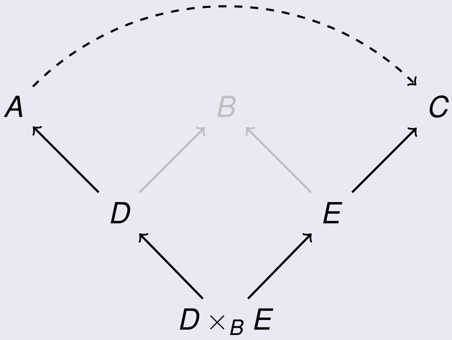

$\rightsquigarrow$ Category of generalized morphisms $\mathbf{G}(\mathbf{A})$

Pseudo-inverses

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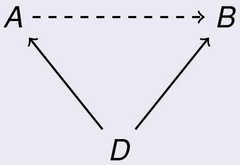

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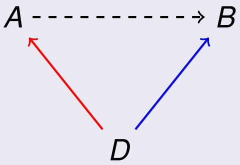

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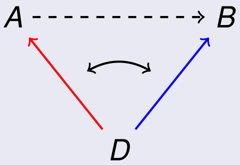

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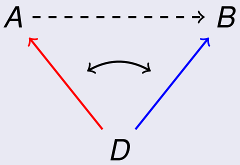

$\rightsquigarrow$

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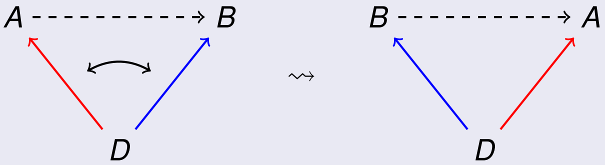

Honest morphisms

Honest morphisms

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$\mathbf{A}$ embeds into $\mathbf{G}(\mathbf{A})$:

Honest morphisms

Honest morphisms

A embeds into **G**(**A**):

$$A \longrightarrow B$$

Honest morphisms

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$$A \longrightarrow B \quad \mapsto$$

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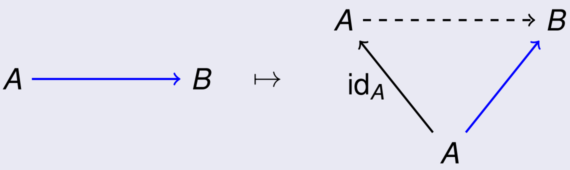

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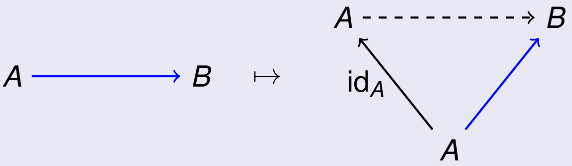

Generalized morphisms equal to such a span are called **honest.**

Computing representatives

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Given an honest generalized morphism in **G**(**A**), compute the corresponding morphism in **A**.

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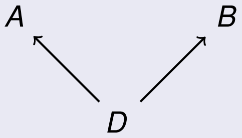

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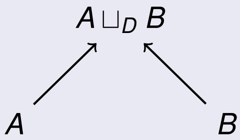

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$$A \longleftarrow A \times_{A \sqcup_D B} B \longrightarrow B$$

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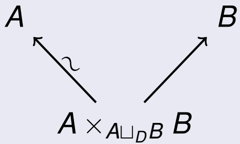

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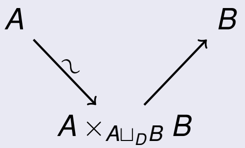

Computing representatives

Given an honest generalized morphism in $\mathbf{G}(\mathbf{A})$, compute the corresponding morphism in $\mathbf{A}$.

$$A \longrightarrow B$$

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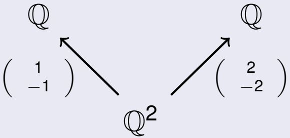

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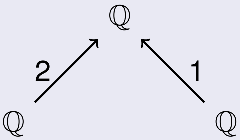

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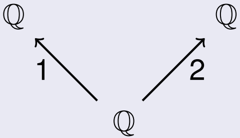

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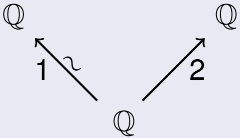

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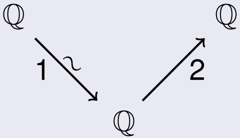

Computing representatives

Given an honest generalized morphism in $\mathbf{G}(\mathbf{A})$, compute the corresponding morphism in $\mathbf{A}$.

$$\mathbb{Q} \xrightarrow{\quad 2 \quad} \mathbb{Q}$$

Constructive diagram chases

Constructive diagram chases

Strategy for constructive diagram chases

Constructive diagram chases

Strategy for constructive diagram chases

1. Compute in $\mathbf{G}(\mathbf{A})$ using pseudo-inverses and compositions.

Constructive diagram chases

Strategy for constructive diagram chases

1. Compute in $\mathbf{G}(\mathbf{A})$ using pseudo-inverses and compositions.
2. Compute the honest representative of the resulting generalized morphism.

Section 3

Constructive spectral sequences

Spectral sequences of bicomplexes

Spectral sequences of bicomplexes

- Every cohomological bicomplex gives rise to a cochain complex, its **total cochain complex.**

Spectral sequences of bicomplexes

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- The total cochain complex admits canonical filtrations.

Spectral sequences of bicomplexes

- Every cohomological bicomplex gives rise to a cochain complex, its **total cochain complex**.
- The total cochain complex admits canonical filtrations.
- We can compute the associated spectral sequences.

Spectral sequences of bicomplexes

Constructing a generalized morphism $C^{p,q+1} \dashrightarrow C^{p+3,q-1}$

$$\begin{array}{ccccccccccc}
\cdots & & \cdots & & \cdots & & \cdots & & \cdots & & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & C^{p,q+1} & \longrightarrow & C^{p+1,q+1} & \longrightarrow & C^{p+2,q+1} & \longrightarrow & C^{p+3,q+1} & \rightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
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 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & C^{p,q+1} & {\color{red}\longrightarrow} & C^{p+1,q+1} & \longrightarrow & C^{p+2,q+1} & \longrightarrow & C^{p+3,q+1} & \rightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
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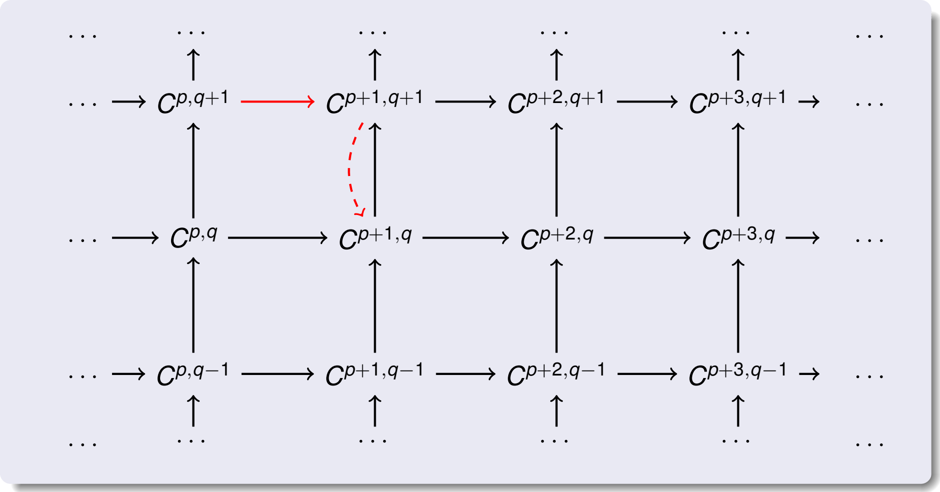

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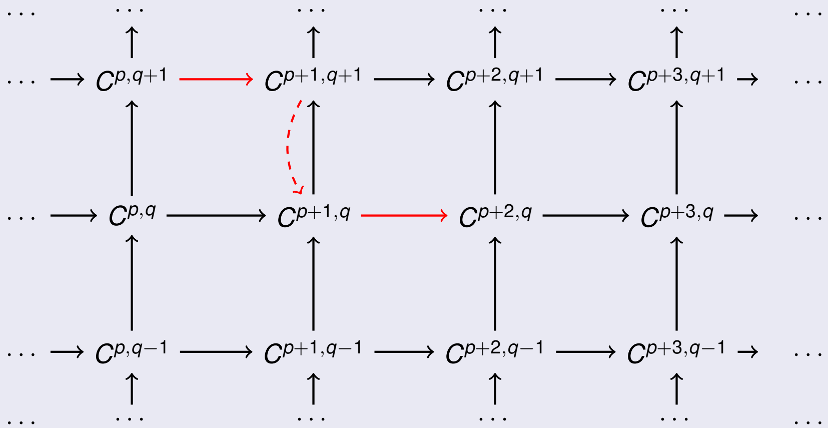

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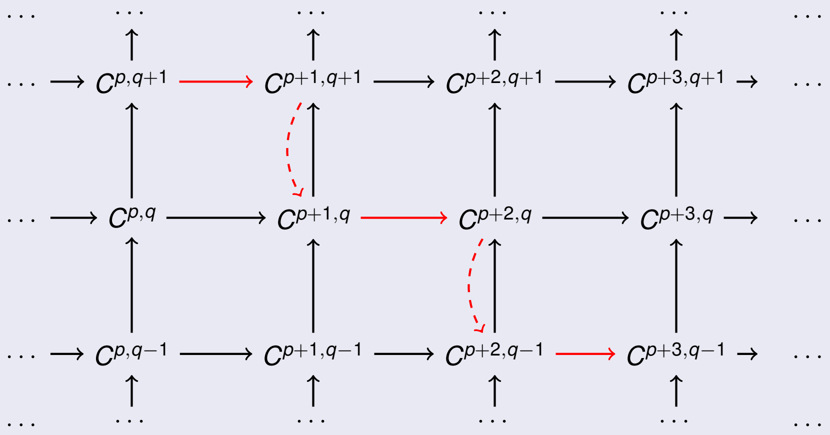
\cdots & \longrightarrow & C^{p,q+1} & {\color{red}\longrightarrow} & C^{p+1,q+1} & \longrightarrow & C^{p+2,q+1} & \longrightarrow & C^{p+3,q+1} & \longrightarrow & \cdots \\
 & & \uparrow & & {\color{red}\downarrow}\,\uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & C^{p,q} & \longrightarrow & C^{p+1,q} & {\color{red}\longrightarrow} & C^{p+2,q} & \longrightarrow & C^{p+3,q} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & {\color{red}\downarrow}\,\uparrow & & \uparrow & & \\
\cdots & \longrightarrow & C^{p,q-1} & \longrightarrow & C^{p+1,q-1} & \longrightarrow & C^{p+2,q-1} & \longrightarrow & C^{p+3,q-1} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & & \cdots & & \cdots & & \cdots & & \cdots & & \cdots
\end{array}
$$

Spectral sequences of bicomplexes

Constructing a generalized morphism $C^{p,q+1} \dashrightarrow C^{p+3,q-1}$



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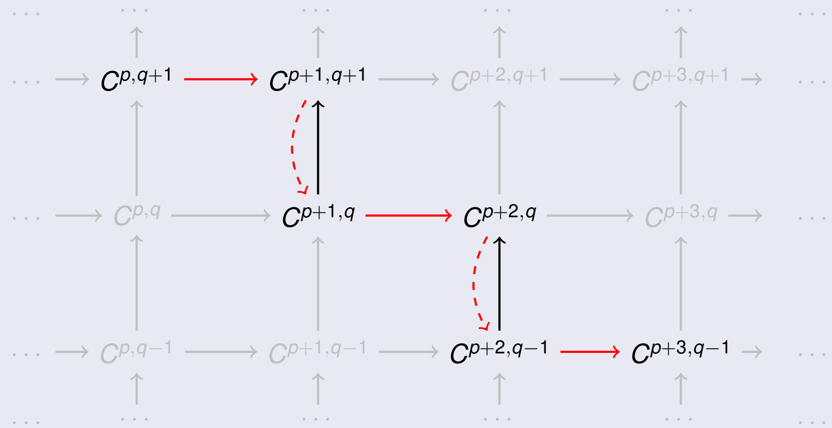

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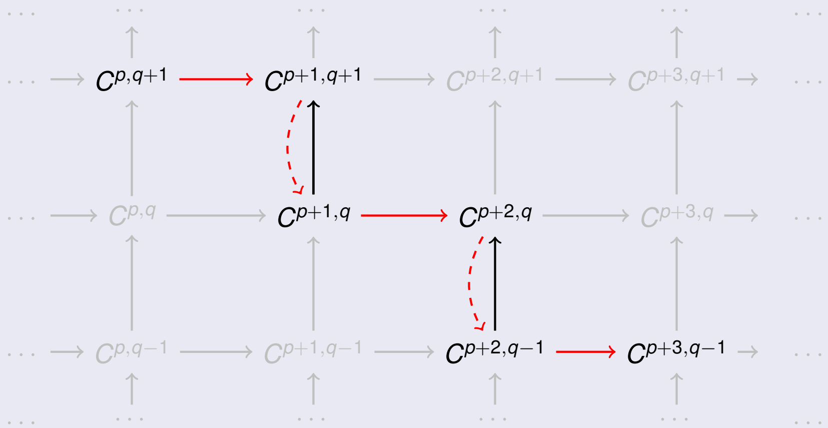

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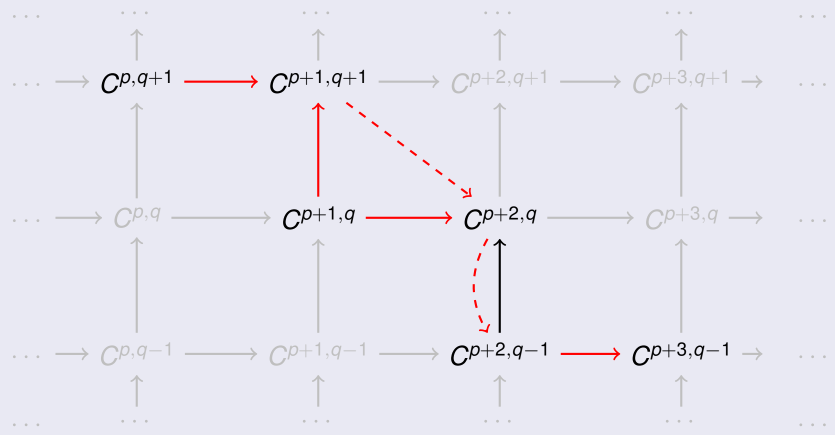

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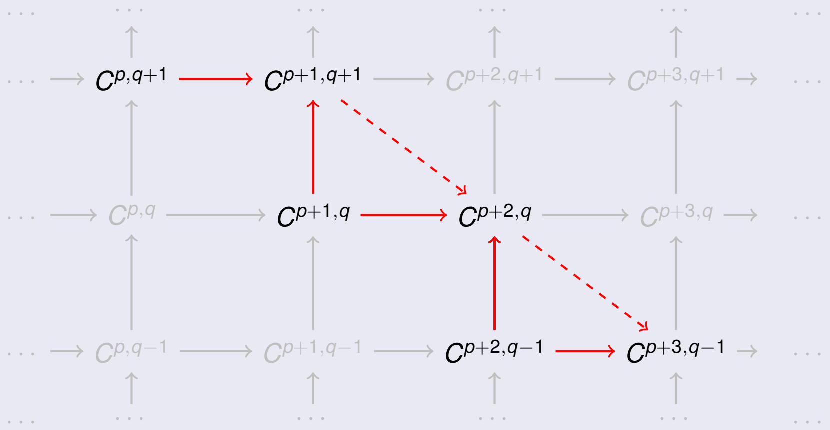

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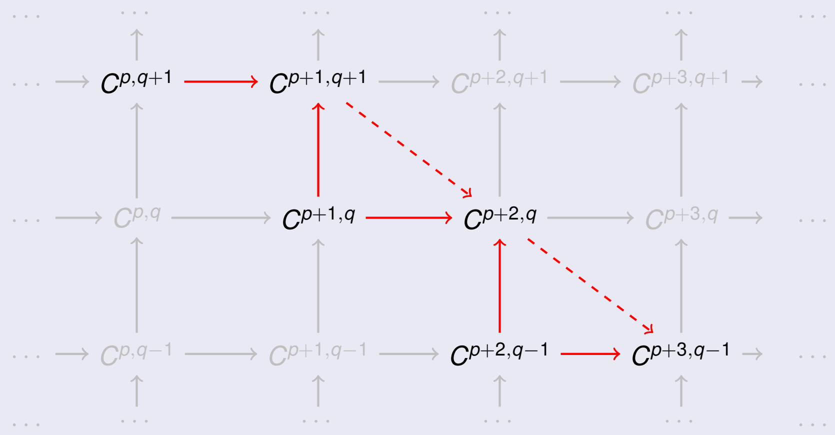

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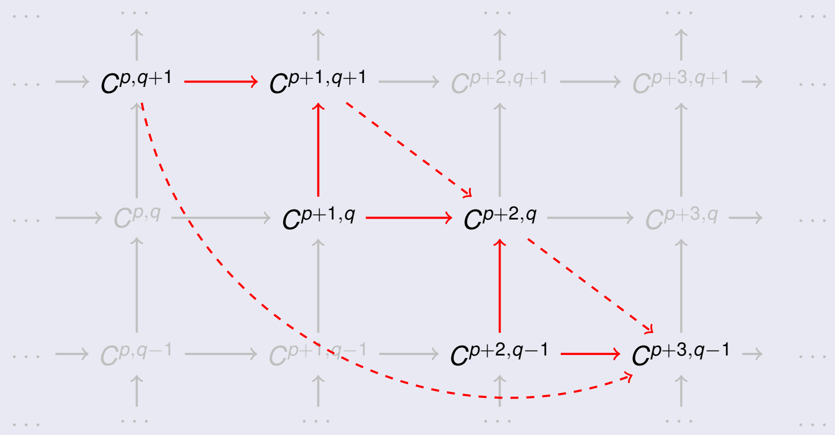


Spectral sequences for bicomplexes

Spectral sequences for bicomplexes

A closed formula for the differentials:

Spectral sequences for bicomplexes

A closed formula for the differentials:

$$C^{p,q+1} \dashrightarrow C^{p+3,q+1}$$

Spectral sequences for bicomplexes

A closed formula for the differentials:

$$E_3^{p,q+1} \hookrightarrow C^{p,q+1} \dashrightarrow C^{p+3,q+1}$$

Spectral sequences for bicomplexes

A closed formula for the differentials:

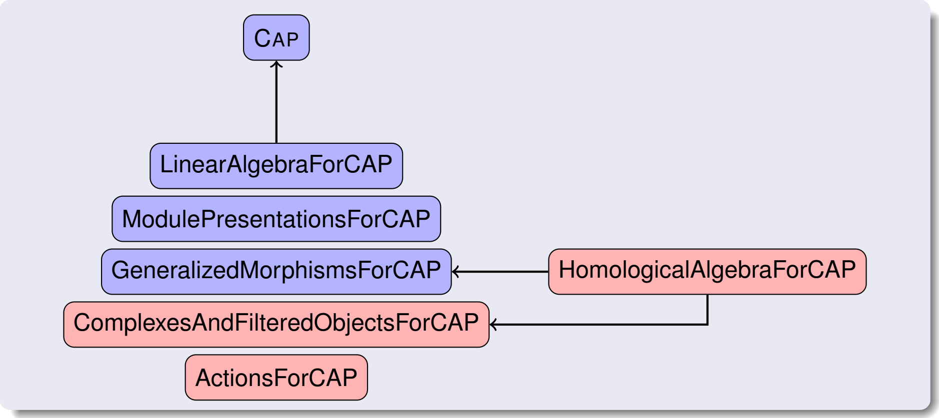
$$E_3^{p,q+1} \hookrightarrow C^{p,q+1} \dashrightarrow C^{p+3,q+1} \twoheadrightarrow E_3^{p+3,q+1}$$

Spectral sequences for bicomplexes

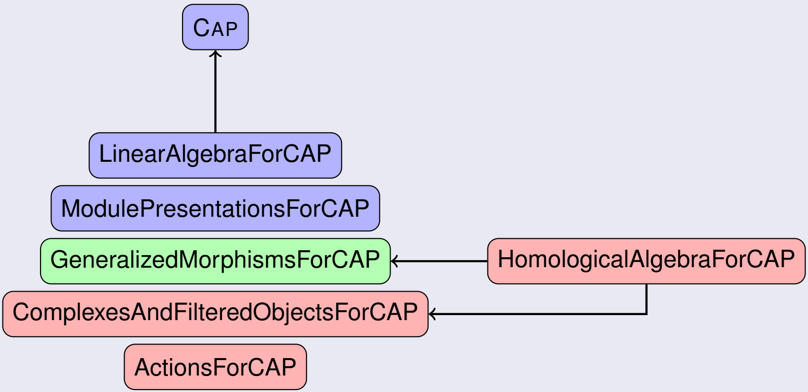
A closed formula for the differentials:

$$d_3^{p,q+1} : E_3^{p,q+1} \hookrightarrow C^{p,q+1} \dashrightarrow C^{p+3,q+1} \twoheadrightarrow E_3^{p+3,q+1}$$

CAP Packages



CAP Packages



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